TECHNICAL REPORT NO. 5-68

THE THEORY OF STRESS DETERMINATION BY MEANS OF STRESS RELIEF TECHNIQUES IN A TRANSVERSELY ISOTROPIC MEDIUM

BY

DENNIS S. BERRY

OCTOBER 1968

MISSOURI RIVER DIVISION, CORPS OF ENGINEERS
OMAHA, NEBRASKA 68101

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WITH CHARLES FAIRHURST, MINNEAPOLIS, MINNESOTA

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Conducted for

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Under

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with

CHARLES FAIRHURST, MINNEAPOLIS, MINNESOTA

FOREWORD

This report was prepared by Dr. Dennis S. Berry under the direction of Charles Fairhurst of the School of Mineral and Metallurgical Engineering, University of Minnesota, Minneapolis, Minnesota under Contract DA-25-066-ENG 14,765 with the Missouri River Division, Corps of Engineers.

Dr. Dennis S. Berry of the University of Nottingham, Nottingham, England was principal investigator for this research.

L. B. Underwood, Division Geologist monitored this research contract for the government; K. S. Lane was Chief of Geology, Soils and Materials Branch, and J. O. Ackerman was Chief of Engineering Division of the Missouri River Division, Corps of Engineers during this investigation.

Funds were provided by the Office, Chief of Engineers, Department of the Army under Military Construction and Investigational Programs, O&M, 7.80.12A QB-1-02-007.

Based on the findings of this report continuing research is being sponsored by Army O&M funds.

PREFACE

This report is part of a continuing study into Methods of Determining In-Situ Rock Stresses at Great Depths. Results of the first part of the study were reported in Technical Report No. 1-68, bearing the title underlined above and published in February, 1968 by the Missouri River Division, Corps of Engineers, Omaha. In that report it was noted that all current techniques of in-situ rock stress determination were based on the assumption that the rock behaved as a linearly elastic, isotropic, continuum. Since many rocks are not isotropic it was felt that an analysis of the influence of rock anisotropy on the accuracy of the techniques was needed. It was therefore decided to attempt solutions to problems in which the rock was considered to behave as a transversely isotropic elastic material. A transversely isotropic material is one for which the elastic properties are invariant with respect to rotations about a single axis only in the material. Five independent elastic constants (see Technical Report I-68 p. 6) are necessary to define a transversely isotropic material compared to two for isotropic material. Bedded or laminated rocks such as shales and gneisses appear to be fairly well described by the transversely isotropic model. The use of more sophisticated models, involving a greater number of physical constants, was considered unwarranted since experimental determination of the constants would be difficult and the mathematics quickly becomes intractable.

The theoretical solution for the Stresses on the Surface of a Circular Hole in an Infinite Transversely Isotropic Elastic Medium due to General Stresses at Infinity and Hydrostatic Pressure at the Hole, was obtained by Dr. D. S. Berry, Department of Theoretical and Applied Mechanics, University of Nottingham, Nottingham, England, and is presented in Appendix 1 of Technical Report I-68. The solution enables the influence of rock anisotropy to be considered in the analysis of the hydraulic fracturing technique of stress determination.

This report, The Theory of Stress Determination by Means of Stress Relief Techniques in a Transversely Isotropic Medium, also by Dr. Berry, presents the analytical expressions for the strains and displacements at the surface of a circular hole in an infinite transversely isotropic elastic medium due to general stresses at infinity. This solution permits the influence of rock anisotropy to be considered in the stress-relief (overcoring) techniques.

Investigation of the effect of rock anisotropy on stresses in an elastic inclusion, the theoretical basis of the remaining important class of stress-determination methods, is now in progress and will be published in a subsequent report.

C. Fairhurst
September 6, 1968

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THE THEORY OF STRESS DETERMINATION BY MEANS OF STRESS RELIEF TECHNIQUES IN A TRANSVERSELY ISOTROPIC MEDIUM

D. S. Berry

1. Introduction

The use of the stress relief technique requires theoretical knowledge of the radial displacement in a long hole as the stress is removed by overcoring or other means. Panek (1966) has pointed out that relief of stress in the axial direction affects the radial displacement in an isotropic medium, while Berry and Fairhurst (1966) have incorporated this effect in calculating results for a transversely isotropic medium in which a hole is drilled in the plane of symmetry or normal to it. This report extends those results to an arbitrarily oriented hole drilled into a medium with any homogeneous state of stress.

The author's previous work on the stress around a hole in a transversely isotropic medium (Appendix I of Tech. Rpt. No. 1-68 by C. Fairhurst, referred to here as I) is drawn upon freely and the same notation is adopted. The method of solution is based upon the work of Milne-Thomson (1962) on "anti-plane" strain.

2. Displacements due to antiplane strain

As in I, the plane $z = 0$ defines an arbitrary cross-section of the hole, the x-axis is chosen to lie in a plane of elastic symmetry (the lines of intersection of these planes with the plane of the paper are indicated in Figs. 1 and 3) in such a way that the angle ϕ that the plane makes with the y-axis satisfies the condition $0 \leq \phi \leq \pi/2$

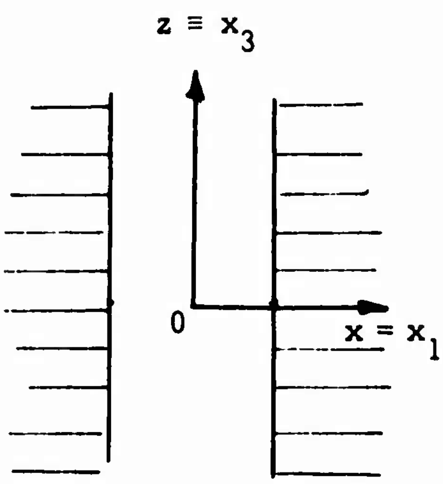

Fig. 1

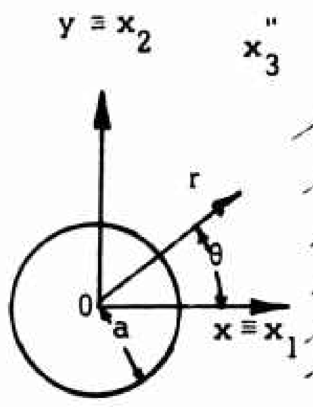

Fig. 2

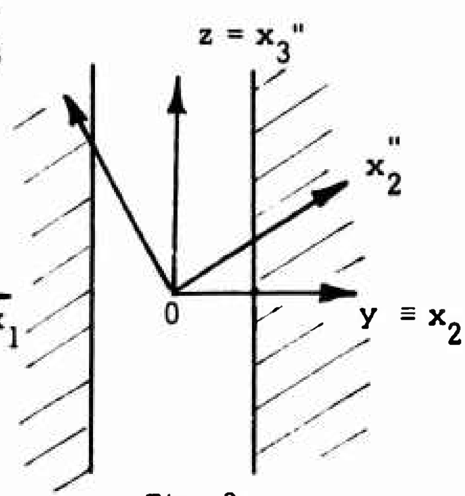

Fig. 3

The coordinate system (x,y,z) is also referred to as (x_1,x_2,x_3). The $0x_3''$ direction is defined as the axis of elastic symmetry (Fig. 3) and the direction cosines, $a_{ij} = \cos(x_i 0 x_j'')$ of the x_i - coordinates with respect to the x_j''-coordinates are given by I(1.1). The stress-strain relations in the x_j''-system are given by I(1.2) and I(1.3), while the stress-strain relations in the x_i - system are given by I(1.5) and I(1.6). The regional stress tensor is given in terms of the principal stresses by I(1.4).

Section 2 of I shows how the stress distribution for an antiplane problem can be obtained in terms of three analytic functions

$$W_\nu(z_\nu) , \qquad \nu = 1,2,3$$

$$z_\nu = x + \lambda_\nu y ,$$

where the λ_ν are roots of the characteristic equation $f(\lambda) = 0$ given by I(2.17) and I(2.18). The stress components are actually given by the expressions I(2.23): -

$$\sigma_x = \frac{1}{4} \sum_{\nu=1}^{3} [\lambda_\nu^2 W_\nu(z_\nu) + \bar{\lambda}_\nu^2 \bar{W}_\nu(\bar{z}_\nu)] ,$$

$$\sigma_y = \frac{1}{4} \sum_{\nu=1}^{3} [W_\nu(z_\nu) + \bar{W}_\nu(\bar{z}_\nu)] ,$$

$$
\left.\begin{aligned}
\tau_{xy} &= -\tfrac{1}{4}\sum_{\nu=1}^{3} [\, \lambda_\nu W_\nu(z_\nu) + \bar{\lambda}_\nu \bar{W}_\nu(\bar{z}_\nu) \,] \; , \\
\tau_{xz} &= \tfrac{1}{4}\sum_{\nu=1}^{3} [\, \lambda_\nu \mu_\nu W_\nu (z_\nu) + \bar{\lambda}_\nu \bar{\mu}_\nu \bar{W}_\nu (\bar{z}_\nu) \,] \; , \\
\tau_{yz} &= -\tfrac{1}{4}\sum_{\nu=1}^{3} [\, \mu_\nu W_\nu (z_\nu) + \bar{\mu}_\nu \bar{W}_\nu (\bar{z}_\nu) \,] \; ,
\end{aligned}\right\} \qquad (2.1)
$$

while σ_z is obtained from equation I(2.5) :

$$
\epsilon_z = k_{13}\sigma_x + k_{23}\sigma_y + k_{33}\sigma_z + k_{34}\tau_{yz} \, . \qquad (2.2)
$$

The constants μ_1 , μ_2 , μ_3 are given by I(2.20).

Substitution of the expressions (2.1) and (2.2) in the relations I(2.6) (valid for <u>constant</u> ϵ_z) gives the following expressions for the strain components:

$$
\epsilon_x = \sum_{\nu=1}^{3} [\, L_{1\nu} W_\nu (z_\nu) + \bar{L}_{1\nu} \bar{W}_\nu (\bar{z}_\nu) \,] + K_1 \epsilon_z \qquad (2.3)
$$

$$
\epsilon_y = \sum_{\nu=1}^{3} [\, L_{2\nu} W_\nu (z_\nu) + \bar{L}_{2\nu} \bar{W}_\nu (\bar{z}_\nu) \,] + K_2 \epsilon_z \qquad (2.4)
$$

$$
2\gamma_{xy} = \sum_{\nu=1}^{3} [\, L_{6\nu} W_\nu (z_\nu) + \bar{L}_6 \bar{W}_\nu (\bar{z}_\nu) \,] \; , \qquad (2.5)
$$

$$
2\gamma_{yz} = \sum_{\nu=1}^{3} [\, L_{4\nu} W_\nu (z_\nu) + \bar{L}_{4\nu} \bar{W}_\nu (\bar{z}_\nu) \,] + K_4 \epsilon_z , \qquad (2.6)
$$

$$2\gamma_{xz} = \sum_{\nu=1}^{3} [\, L_{5\nu} W_\nu(z_\nu) + \bar{L}_{5\nu} \bar{W}_\nu(\bar{z}_\nu) \,] \,, \qquad (2.7)$$

where

$$4L_{1\nu} = \ell_{11}\lambda_\nu^2 + \ell_{12} - \ell_{14}\mu_\nu \,,$$

$$4L_{2\nu} = \ell_{12}\lambda_\nu^2 + \ell_{22} - \ell_{24}\mu_\nu \,,$$

$$4L_{6\nu} = -\ell_{66}\lambda_\nu + \ell_{56}\lambda_\nu\mu_\nu \,, \qquad (2.8)$$

$$4L_{4\nu} = \ell_{14}\lambda_\nu^2 + \ell_{24} - \ell_{44}\mu_\nu \,,$$

$$4L_{5\nu} = -\ell_{56}\lambda_\nu + \ell_{55}\lambda_\nu\mu_\nu \,,$$

and the ℓ_{rs} and K_j are given by I (2.6^1).

If we write $V_\nu(z_\nu) = \int W_\nu(z_\nu)\,dz_\nu$, then integration of (2.3) with respect to x and (2.4) with respect to y yields the following two expressions for components of displacement

$$u = \sum_{\nu=1}^{3} [\, L_{1\nu} V_\nu(z_\nu) + \bar{L}_{1\nu} \bar{V}_\nu(\bar{z}_\nu) \,] + K_1 \varepsilon_z x + f_1(y) \qquad (2.9)$$

$$v = \sum_{\nu=1}^{3} \left[\frac{L_{2\nu}}{\lambda_\nu} V_\nu(z_\nu) + \frac{\bar{L}_{2\nu}}{\bar{\lambda}_\nu} \bar{V}_\nu(\bar{z}_\nu) \right] + K_2 \varepsilon_z y + f_2(x) \qquad (2.10)$$

where $f_1(y)$ and $f_2(x)$ are arbitrary functions. An expression for $2\gamma_{xy} = \partial u/\partial y + \partial v/\partial x$ can be found from (2.9) and (2.10) and comparison with (2.5) shows that $f_1(y)$ and $f_2(x)$ are constants (which may be taken as zero since they correspond to rigid body motion only).

Since we are considering deformations which are independent of z, we have that

$$\partial v/\partial z \equiv \partial u/\partial z \equiv 0$$

and so

$$2\gamma_{yz} = \frac{\partial w}{\partial y} \quad \text{and} \quad 2\gamma_{xz} = \frac{\partial w}{\partial x} .$$

Then integration of (2.6) and (2.7) give

$$w = \sum_{\nu=1}^{3} \left[\frac{L_{4\nu}}{\lambda_\nu} V_\nu(z_\nu) + \frac{\bar{L}_{4\nu}}{\bar{\lambda}_\nu} \bar{V}_\nu(\bar{z}_\nu) \right] + K_4\epsilon_z y + f_3(x) \qquad (2.11)$$

and

$$w = \sum_{\nu=1}^{3} \left[L_{5\nu} V_\nu(z_\nu) + \bar{L}_{5\nu} \bar{V}_\nu(\bar{z}_\nu) \right] + f_4(y). \qquad (2.12)$$

Hence, apart from constants, which may be ignored

$$f_3(x) \equiv 0 \ , \quad f_4(y) = K_4\epsilon_z y .$$

Also, we find that

$$L_{5\nu} - \frac{L_{4\nu}}{\lambda_\nu} = \frac{d^2(\lambda_\nu)}{4\lambda_\nu} \left[\mu_\nu - \frac{d^{(3)}(\lambda_\nu)}{d^{(2)}(\lambda_\nu)} \right]$$

where $d^2(\lambda)$ and $d^3(\lambda)$ are given by I(2.18) and this is identically zero by the definition of μ_ν in I(2.20).

Collecting the above results we have the following expressions for the three components of displacement:

$$u = \sum_{\nu=1}^{3} \left[L_{1\nu} V_\nu(z_\nu) + \bar{L}_{1\nu} \bar{V}_\nu(\bar{z}_\nu) \right] + K_1\epsilon_z x ,$$

$$v = \sum_{\nu=1}^{3} \left[\frac{L_{2\nu}}{\lambda_\nu} V_\nu(z_\nu) + \frac{\bar{L}_{2\nu}}{\bar{\lambda}_\nu} \bar{V}_\nu(\bar{z}_\nu) \right] + K_2\epsilon_z y ,$$

$$w = \sum_{\nu=1}^{3} \left[\frac{L_{4\nu}}{\lambda_\nu} V_\nu (z_\nu) + \frac{\bar{L}_{4\nu}}{\bar{\lambda}_\nu} \bar{V}_\nu (\bar{z}_\nu) \right] + K_4 \varepsilon_z y \, . \qquad (2.13)$$

3. Formulation of the problem.

In the stress-relief method, diametral displacement indicators (possibly strain-gauges also) are inserted at a section of a hole sufficiently remote from its ends for their influence to be negligible, the surrounding stress is relieved and changes in diametral measurements of the hole are recorded, possibly changes in axial strain also. If the regional stress components are denoted by $(\sigma_{ij})_o$, as in Section 3 of I, we can describe their removal as the application of stress components $-(\sigma_{ij})_o$, and the boundary conditions can be stated as follows:

$$\left. \begin{aligned} \sigma_r &= \tau_{r\theta} = \tau_{rz} = 0 \, , \quad \text{on } r = a \, , \\ \sigma_{ij} &= -(\sigma_{ij})_o \quad \text{at } \infty \, . \end{aligned} \right\} \qquad (3.1)$$

Now these conditions are just those of I(3.1) apart from the minus sign in front of $(\sigma_{ij})_o$. In addition, it is specified in I Section 3 that ε_z should be constant everywhere: if we can show that that condition is satisfied in the stress-relief problem then the stress functions for the solution of I(3.1) can be adopted for the present problem, merely with a change of sign. It can be demonstrated as follows.

Since we are assuming that the material conforms to the assumptions of classical elasticity we can invoke the principle of superposition and, in particular, we know that the chronological order in which two or more constraints are imposed does not affect the resulting elastic field. Consequently, the axial strain in a stress-relieved sample is the same as if the hole were made after removal of the sample instead of before. Now, at a cross section remote from the ends we can safely assume that the in situ drilling produces no axial strain. Also, drilling a hole after removal produces no axial strain because the sample now has no applied stresses capable of producing any deformation. However, the final state is the same in each case, so the axial strain is just that which is produced by relieving stress in the sample without a hole. Thus the axial strain is not dependent on the presence of the hole and so is independent of position in the rock.

If we write out the second set of conditions in (3.1) in non-tensor notation, using $\sigma_1, \ldots. \tau_{12}, \ldots$ for the regional stress components, we have at ∞

$$\sigma_x = -\sigma_1, \qquad \sigma_y = -\sigma_2, \qquad \sigma_z = -\sigma_3,$$

$$\tau_{xy} = -\tau_{12}, \qquad \tau_{yz} = -\tau_{23}, \qquad \tau_{zx} = -\tau_{31}.$$

If these values are substituted in (2.2) we obtain the axial strain in terms of the regional stress components, valid for <u>all</u> x and y since ϵ_z is constant:

$$\epsilon_z = -(k_{13}\sigma_1 + k_{23}\sigma_2 + k_{33}\sigma_3 + k_{34}\tau_{23}). \tag{3.2}$$

Elimination of ϵ_z between (2.2) and (3.2) leads to the condition

$$k_{13}(\sigma_1 + \sigma_x) + k_{23}(\sigma_2 + \sigma_y) + k_{33}(\sigma_3 + \sigma_z) + k_{34}(\tau_{23} + \tau_{yz}) = 0 \tag{3.3}$$

which is just I(3.2), apart from the change of sign giving $\sigma_x = -(\sigma_x)_1$ etc.

4. Solution of the general problem.

The conditions to be satisfied are the values of the boundary tractions and stress components at infinity, given by (3.1), together with the general condition on the stress components given by (3.3).

The complex potential method gives stress solutions in the form (2.1), apart from the component σ_z which can be obtained from relation (3.3).

The solution of I(3.1) under condition I(3.2) gives stress components denoted by $(\sigma_x)_1$ etc. in I and is obtained from the complex potentials denoted by $[W_\nu(z_\nu)]_1$. Clearly the solution given by complex potentials

$$W_\nu(z_\nu) = -[\,W_\nu(z_\nu)\,]_1 \tag{4.1}$$

will give stress components $\sigma_x = -(\sigma_x)_1$ everywhere and $\sigma_{ij} = -(\sigma_{ij})_o$ at infinity in particular, thus satisfying the conditions (3.1) and (3,3).

Thus, from Section 4 of I the stress-relief solution is given by (3.2) and (3.3) and the expressions

$$-W_\nu(z_\nu) = a_{\nu o} + W_\nu^*(z_\nu)\,, \qquad \nu = 1,2,3, \tag{4.2}$$

$$W_\nu^*(z_\nu) = \frac{\delta_{\nu\nu}C_1 + \gamma_{\nu\nu}C_2 + \mu_{\nu\nu}C_3}{2\zeta^2 m_\nu'(\zeta)} \tag{4.3}$$

$$z_\nu = m_\nu(\zeta) = a\,(\gamma_\nu\,\zeta + \delta_\nu/\zeta)\,, \tag{4.4}$$

or

$$\zeta = \frac{z_\nu + [\,z^2 - \alpha_\nu^2\,]^{\frac{1}{2}}}{2a\,\gamma_\nu} \tag{4.5}$$

The various constants in (4.2) to (4.5) are given by equations I(4.6) to I (4.9), with the addition

$$\alpha_\nu^2 = a^2(1 + \lambda_\nu^2).$$

Now, from I(4.11) we have that

$$\delta_{\nu\nu}C_1 + \gamma_{\nu\nu}C_2 + \mu_{\nu\nu}C_3 = 2a(a_{\nu o}\,\delta_\nu + A_\nu) \tag{4.6}$$

with the A_ν given by I(4.12) , while it is easy to show that

$$\zeta\, m_\nu'(\zeta) = [\,z_\nu^2 - a^2(1 + \lambda_\nu^2)\,]^{\frac{1}{2}}. \tag{4.7}$$

Substitution of (4.6) and (4.7) into (4.3) leads to the expression

$$W^*_\nu(z_\nu) = \frac{a_{\nu o}\delta_\nu + A_\nu}{2\delta_\nu}\left[\frac{z_\nu}{(z_\nu^2 - \alpha_\nu^2)^{\frac{1}{2}}} - 1\right]. \tag{4.8}$$

From (4.2) we then find the result

$$-W_\nu(z_\nu) = \tfrac{1}{2}(a_{\nu o} - \frac{A_\nu}{\delta_\nu}) + \tfrac{1}{2}(a_{\nu o} + \frac{A_\nu}{\delta_\nu})\frac{z_\nu}{(z_\nu^2 - \alpha_\nu^2)^{\frac{1}{2}}} \tag{4.9}$$

and so by integration we find that

$$-V_\nu(z_\nu) = -\int W_\nu(z_\nu)\,dz_\nu = \tfrac{1}{2}(a_{\nu o} - \frac{A_\nu}{\delta_\nu})\,z_\nu + \tfrac{1}{2}(a_{\nu o} + \frac{A_\nu}{\delta_\nu})(z_\nu^2 - \alpha_\nu^2)^{\frac{1}{2}} \tag{4.10}$$

For the stress-relief problem we require the radial displacement at the boundary of the hole, $r = a$. At the boundary

$$z_\nu = x + \lambda_\nu y = a(\cos\theta + \lambda_\nu \sin\theta)$$

and

$$z_\nu^2 - \alpha_\nu^2 = a^2(\cos\theta + \lambda_\nu \sin\theta)^2 - a^2(1 + \lambda_\nu^2)$$

$$= -a^2(\sin\theta - \lambda_\nu \cos\theta)^2$$

so that

$$(z_\nu^2 - \alpha_\nu^2)^{\frac{1}{2}} = i\,a(\sin\theta - \lambda_\nu \cos\theta).$$

(The choice of sign here is immaterial because of the symmetry of the solution).

Substitution in (4.10) leads to the result for $r = a$

$$V_\nu(z_\nu) = -a(a_{\nu o}\gamma_\nu e^{i\theta} - A_\nu e^{-i\theta}), \tag{4.11}$$

or

$$V_\nu(z_\nu) = -a[(a_{\nu o}\gamma_\nu - A_\nu)\cos\theta + i(a_{\nu o}\gamma_\nu + A_\nu)\sin\theta]. \tag{4.12}$$

To obtain the radial displacement from these expressions and the displacement expressions (2.13) we note first that

$$u_r + iu_\theta = e^{-i\theta}(u + iv) . \tag{4.13}$$

From (2.13) it then follows that

$$u_r + iu_\theta = e^{-i\theta}\left\{ \sum_{\nu=1}^{3} \left[\left(L_{1\nu} + i\frac{L_{2\nu}}{\lambda_\nu}\right) V_\nu(z_\nu) + \left(\bar{L}_{1\nu} + i\frac{\bar{L}_{2\nu}}{\bar{\lambda}_\nu}\right) \bar{V}_\nu(\bar{z}_\nu) \right] \right.$$

$$\left. + a\,\epsilon_z\,(K_1\cos\theta + iK_2\sin\theta) \right\} . \tag{4.14}$$

Use of (4.11) then gives the result

$$u_r + iu_\theta = -a\left\{ \sum_{\nu=1}^{3} \left[\left(L_{1\nu} + i\frac{L_{2\nu}}{\lambda_\nu}\right)\left(a_{\nu o}\gamma_\nu - A_\nu e^{-2i\theta}\right) \right. \right.$$

$$\left. + \left(\bar{L}_{1\nu} + i\frac{\bar{L}_{2\nu}}{\bar{\lambda}_\nu}\right)\left(\bar{a}_{\nu o}\bar{\gamma}_\nu e^{-2i\theta} - \bar{A}_\nu\right) \right]$$

$$\left. + \epsilon_z e^{-i\theta}(K_1\cos\theta + iK_2\sin\theta) \right\} . \tag{4.15}$$

We now put

$$G_\nu = L_{1\nu} + i\frac{L_{2\nu}}{\lambda_\nu} , \quad H_\nu = L_{1\nu} - i\frac{L_{2\nu}}{\lambda_\nu} \tag{4.16}$$

and take the real part of (4.15) to obtain

$$\frac{u_r}{a} = A_\phi + B_\phi\cos 2\theta + C_\phi\sin 2\theta , \tag{4.17}$$

where

$$A_\phi = \mathrm{Re}\sum_{\nu=1}^{3}\left(A_\nu H_\nu - a_{\nu o}\gamma_\nu G_\nu\right) + \tfrac{1}{2}\epsilon_z(K_1 + K_2) ,$$

$$B_\phi = \mathrm{Re}\sum_{\nu=1}^{3}\left(A_\nu G_\nu - a_{\nu o}\gamma_\nu H_\nu\right) + \tfrac{1}{2}\epsilon_z(K_1 - K_2) , \tag{4.18}$$

$$C_{\phi} = \mathrm{Im} \sum_{\nu=1}^{3} (A_{\nu} G_{\nu} + a_{\nu 0} \gamma_{\nu} H_{\nu}) .$$

Equations (4.17) and (4.18) give the diametral expansion of the hole (when the regional stress is relieved) in terms of the regional stress components (through the $a_{\nu 0}$ and ϵ_z) and the elastic constants of the medium. The orientation of the hole, ϕ , determines the values of the constants which actually appear in the above equations, so that the values of A_{ϕ} , B_{ϕ} and C_{ϕ} are dependent on ϕ as well as on the regional stress components for a given rock mass.

5. The exceptional cases.

As the author explained in I, the cases $\phi = 0, \pi/2$ are not covered by the general solution for antiplane strain in a transversely isotropic medium because the stress functions χ (x,y) and ψ (x,y) then satisfy independent equations, instead of the mixed equations I(2.9) and I(2.10). The result is that components of displacement u and v (or u_r and u_{θ}) are independent of the regional stress components τ_{13} and τ_{23}.

The required results for radial deformation due to stress relief have been given by Berry and Fairhurst (1966) in terms of principal stress components assumed to lie parallel to the axis of the hole and normal to it. The derivation of these results is given in this section, using the notation of this report.

The result for an isotropic material can be easily obtained from either of these results.

5.1 Case $\phi = 0$

We first show the close correspondence between plane-strain problems in this case and similar problems in an isotropic material. This enables us to use the well known plane-strain result for a stress-free hole in an isotropic material. The correction for non-zero axial strain is then applied to complete the solution.

From I(1.5) and I(5.1), the stress-strain relations for $\phi = 0$ reduce to

$$
\begin{aligned}
\epsilon_x &= b_{11}\sigma_x + b_{12}\sigma_y + b_{13}\sigma_z \,, \\
\epsilon_y &= b_{12}\sigma_x + b_{11}\sigma_y + b_{13}\sigma_z \,, \\
\epsilon_z &= b_{13}\sigma_x + b_{13}\sigma_y + b_{33}\sigma_z \,, \\
2\gamma_{yz} &= b_{44}\tau_{yz} \,, \\
2\gamma_{zx} &= b_{44}\tau_{zx} \,, \\
2\gamma_{xy} &= 2(b_{11} - b_{12})\,\tau_{xy} \,,
\end{aligned}
\tag{5.1}
$$

where the coefficients are given by I(1.3) in terms of moduli and Poisson's ratios.

For plane-strain, $\epsilon_z = \gamma_{yz} = \gamma_{zx} = 0$ and hence

$$
\sigma_z = -\frac{b_{13}}{b_{33}}\,(\sigma_x + \sigma_y). \tag{5.2}
$$

By substitution from (5.2) the remaining three equations then become

$$
\begin{aligned}
\epsilon_x &= \ell_{11}\sigma_x + \ell_{12}\sigma_y \,, \\
\epsilon_y &= \ell_{12}\sigma_x + \ell_{11}\sigma_y \,, \\
\gamma_{xy} &= (\ell_{11} - \ell_{12})\,\tau_{xy} \,,
\end{aligned}
\tag{5.3}
$$

where

$$\ell_{11} = b_{11} - \frac{b_{13}^2}{b_{33}} = \frac{1}{E_1}\left(1 - \frac{E_2\nu_2^2}{E_1}\right) ,$$

$$\ell_{12} = b_{12} - \frac{b_{13}^2}{b_{33}} = -\frac{1}{E_1}\left(\nu_1 - \frac{E_2}{E_1}\nu_2^2\right) , \tag{5.4}$$

using I(1.3).

Now equations (5.4) have precisely the same form as the plane-strain equations of an isotropic medium. It follows that the same methods of solution are applicable (we have already seen in I that the stress function is biharmonic), and the actual solutions are identical so long as the constants of (5.4) are substituted for the corresponding constants for an isotropic material. The latter are

$$\ell_{11}^I = \frac{1-\nu^2}{E} = \frac{1-\nu}{2\mu} \text{ and } \ell_{12}^I = -\frac{\nu(1+\nu)}{E} = -\frac{\nu}{2\mu} . \tag{5.5}$$

Solving for μ and ν we have

$$\mu = \frac{1}{2(\ell_{11}^I - \ell_{12}^I)} \quad \text{and} \quad \nu = -\frac{\ell_{12}^I}{\ell_{11}^I - \ell_{12}^I} . \tag{5.6}$$

The general procedure is then to take the isotropic solution, express μ and ν in terms of ℓ_{11}^I and ℓ_{12}^I by means of (5.6) and then to substitute the ℓ_{11} and ℓ_{12} of (5.4). The plane-strain solution for relief of regional stress components σ_1, σ_2, τ_{12} around a hole of radius <u>a</u> in an <u>isotropic</u> medium gives a radial displacement of

$$\frac{u_r}{a} = -\frac{1-\nu^2}{E}\left[\sigma_1 + \sigma_2 + 2(\sigma_1 - \sigma_2)\cos 2\theta + 4\tau_{12}\sin 2\theta\right] . \tag{5.7}$$

The factor containing the elastic constants can be replaced directly by ℓ_{11}^I by means of (5.5) and substitution of ℓ_{11} from (5.4) gives, for the <u>transversely isotropic</u> medium

$$\frac{u_r}{a} = -\left(b_{11} - \frac{b_{13}^2}{b_{33}}\right)\left[\sigma_1 + \sigma_2 + 2(\sigma_1 - \sigma_2)\cos 2\theta + 4\tau_{12}\sin 2\theta\right] . \quad (5.8)$$

This result is obtained on the assumption of plane-strain, in particular that $\epsilon_z = 0$. Since that is true only when the stress components are related as in (5.2), a correction term must be added to (5.8) to take account of the more general case. It was noted in Section 3 that ϵ_z is independent of position, so that we may substitute in the third equation of (5.1) the values of the stress components at infinity to determine ϵ_z: that is (since the effective values correspond to removal of the regional stress),

$$\epsilon_z = -(b_{13}\sigma_1 + b_{13}\sigma_2 + b_{33}\sigma_3). \quad (5.9)$$

This strain is due to the fact that the axial stress, σ_z, is greater than that given by the plane-strain condition (5.2) by a quantity σ_z^o (say), and we must now use equations (5.1) to determine σ_z^o and the additional strains ϵ_x^o and ϵ_y^o which it causes.

By putting $\sigma_x = \sigma_y = 0$ in (5.1) we find that

$$\epsilon_x^o = \epsilon_y^o = b_{13}\sigma_z^o ,$$

$$\epsilon_z = b_{33}\sigma_z^o ,$$

and hence

$$\epsilon_x^o = \epsilon_y^o = \frac{b_{13}}{b_{33}}\epsilon_z ,$$

and so, from (5.9), in terms of the regional stresses,

$$\epsilon_x^o = \epsilon_y^o = -b_{13}\sigma_3 - \frac{b_{13}^2}{b_{33}}(\sigma_1 + \sigma_2). \quad (5.10)$$

Since $\epsilon_x^o = \epsilon_y^o$, the additional strain is independent of θ and so induces a diametral deformation of the same value at the hole. Addition of (5.10) to (5.8) leads to the final result for stress-relief:

$$\frac{u_r}{a} = A_o + B_o \cos 2\theta + C_o \sin 2\theta \,, \qquad (5.11)$$

where

$$A_o = -b_{11}(\sigma_1 + \sigma_2) - b_{13}\sigma_3 \,,$$

$$B_o = -2\left(b_{11} - \frac{b_{13}^2}{b_{33}}\right)(\sigma_1 - \sigma_2) \,, \qquad (5.12)$$

$$C_o = -4\left(b_{11} - \frac{b_{13}^2}{b_{33}}\right)\tau_{12} \,,$$

or, from I(1.3):

$$A_o = -\frac{1}{E_1}(\sigma_1 + \sigma_2 - \nu_2\sigma_3) \,,$$

$$B_o = -\frac{2}{E_1}(1 - \nu_2^2)(\sigma_1 - \sigma_2) \,, \qquad (5.13)$$

$$C_o = -\frac{4}{E_1}(1 - \nu_2^2)\,\tau_{12} \,.$$

It should be noted that, for $\phi = 0$, the x and y directions are defined only to the extent that they are normal to the axis of elastic symmetry.

5.2 Case $\phi = \frac{\pi}{2}$.

As noted in I, the plane-strain problem in this case has been discussed by Green and Zerna (1954), although the displacement solution has not been given. However, it is possible to use the complex potentials given by them, as stated in I(5.31) and I(5.32), with the general expression for the complex displacement, to obtain the plane-strain displacement solution. As before, the stress-relief solution is completed by making a correction for the non-zero axial strain.

From I(1.5) and I(5.20), the stress-strain relations for $\phi = \frac{\pi}{2}$ are

$$
\begin{aligned}
\epsilon_x &= b_{11}\sigma_x + b_{13}\sigma_y + b_{12}\sigma_z \,, \\
\epsilon_y &= b_{13}\sigma_x + b_{33}\sigma_y + b_{13}\sigma_z \,, \\
\epsilon_z &= b_{12}\sigma_x + b_{13}\sigma_y + b_{11}\sigma_z \,, \\
2\gamma_{yz} &= b_{44}\tau_{yz} \\
2\gamma_{zx} &= 2(b_{11}-b_{12})\,\tau_{zx} \\
2\gamma_{xy} &= b_{44}\tau_{xy} \,,
\end{aligned}
\qquad (5.14)
$$

expressed in the coefficients of I(1.3).

For plane-strain, $\epsilon_z = \gamma_{yz} = \gamma_{zx} = 0$ and hence

$$
\sigma_z = -\frac{1}{b_{11}}\,(b_{12}\,\sigma_x + b_{13}\sigma_y) \,, \qquad (5.15)
$$

and the remaining equations become

$$
\begin{aligned}
\epsilon_x &= \ell_{11}\sigma_x + \ell_{12}\sigma_y \,, \\
\epsilon_y &= \ell_{12}\sigma_x + \ell_{22}\sigma_y \,, \\
2\gamma_{xy} &= b_{44}\tau_{xy} \,,
\end{aligned}
\qquad (5.16)
$$

where

$$
\ell_{11} = b_{11} - \frac{b_{12}^2}{b_{11}} = \frac{1-\nu_1^2}{E_1} \,,
$$

$$
\ell_{12} = b_{13} - \frac{b_{12}b_{13}}{b_{11}} = -\frac{\nu_2}{E_1}\,(1+\nu_1) \,,
$$

$$\ell_{22} = b_{33} - \frac{b_{13}^2}{b_{11}} = \frac{1}{E_2} - \frac{\nu_2^2}{E_1} \, , \tag{5.17}$$

$$b_{44} = 1/M \, .$$

In the notation of Green and Zerna the complex displacement due to this kind of plane strain deformation in a transversely isotropic medium can be expressed thus:

$$u + iv = \delta_1 \Omega'(z_1) + \rho_1 \overline{\Omega}'(\overline{z}_1) + \delta_2 \omega'(z_2) + \rho_2 \overline{\omega}'(\overline{z}_2) \, , \tag{5.18}$$

where

$$\delta_1 = (1 + \gamma_1)\beta_2 - (1 - \gamma_1)\beta_1 \, ,$$

$$\delta_2 = (1 + \gamma_2)\beta_1 - (1 - \gamma_2)\beta_2 \, ,$$

$$\overline{\rho}_1 = (1 + \gamma_1)\beta_2 + (1 - \gamma_1)\beta_1 \, ,$$

$$\overline{\rho}_2 = (1 + \gamma_2)\beta_1 + (1 - \gamma_2)\beta_2 \, ,$$

$$\gamma_j = \frac{\alpha_j - 1}{\alpha_j + 1} \, , \quad \beta_j = \ell_{12} - \ell_{22}\alpha_j^2 \, , \tag{5.19}$$

$$\alpha_1^2 = \ell_{12} + \tfrac{1}{2} b_{44} + [\, (\ell_{12} + \tfrac{1}{2} b_{44})^2 - \ell_{12}\ell_{22} \,]^{\frac{1}{2}} \, ,$$

$$\alpha_2^2 = \ell_{12} + \tfrac{1}{2} b_{44} - [\, (\ell_{12} + \tfrac{1}{2} b_{44})^2 - \ell_{12}\ell_{22} \,]^{\frac{1}{2}} \, ,$$

and

$$z_j = z + \gamma_j \overline{z} \, , \qquad j = 1, 2 \, .$$

For problems involving circular boundaries the z_j are more conveniently expressed in terms of new variables ζ_1 and ζ_2 such that

$$z_j = \zeta_j + \gamma_j a^2/\zeta_j , \qquad j = 1,2 .$$

and then $\zeta_1 = \zeta_2 = ae^{i\theta}$ on the boundary (when $z = ae^{i\theta}$). The complex potentials for the present problem are given in equations I(5.31) to I(5.33), which are here restated in the form:

$$\Omega'(z_1) = f(\zeta_1) , \quad \omega'(z_2) = g(\zeta_2) , \tag{5.20}$$

where

$$f(\zeta) = A\zeta - [(1-\bar{\gamma}_1\gamma_2)\bar{A} + (1-\gamma_2\bar{\gamma}_2)\bar{A}'] \frac{a^2}{\zeta(\gamma_1-\gamma_2)} ,$$

$$(5.21)$$

$$g(\zeta) = A'\zeta + [(1-\gamma_1\bar{\gamma}_1)\bar{A} + (1-\gamma_1\bar{\gamma}_2)\bar{A}'] \frac{a^2}{\zeta(\gamma_1-\gamma_2)} ,$$

and

$$32A = \frac{(\alpha_1+1)^2(\alpha_2+1)^2}{\alpha_1^2-\alpha_2^2} [(1+\gamma_2)^2(\sigma_1+\sigma_2)+2\gamma_2(\sigma_1-\sigma_2)]$$

$$+ \frac{2i}{\alpha_1}(\alpha_1+1)\tau_{12} ,$$

$$(5.22)$$

$$32A' = -\frac{(\alpha_1+1)^2(\alpha_2+1)^2}{\alpha_1^2-\alpha_2^2} [(1+\gamma_1)^2(\sigma_1+\sigma_2)+2\gamma_1(\sigma_1-\sigma_2)]$$

$$+ \frac{2i}{\alpha_2}(\alpha_2+1)^2\tau_{12} .$$

On the boundary , these equations lead to the result

$$\Omega'(z_1) = f(ae^{i\theta})$$

$$- Aae^{i\theta} - \frac{[(1-\bar{\gamma}_1\gamma_2)\bar{A} + (1-\gamma_2\bar{\gamma}_2)\bar{A}']\, ae^{-i\theta}}{\gamma_1 - \gamma_2},$$

$$\omega'(z_2) = g(ae^{i\theta}) \tag{5.23}$$

$$= A'ae^{i\theta} + \frac{(1-\gamma_1\bar{\gamma}_1)\bar{A} + (1-\gamma_1\bar{\gamma}_2)\bar{A}'}{\gamma_1 - \gamma_2}\, ae^{-i\theta},$$

and so, from (5.18),

$$\frac{u_r + iu_\theta}{a} = e^{-i\theta}\,\frac{(u+iv)}{a}$$

$$= \delta_1 A + \delta_2 A' - \frac{\rho_1[(1-\gamma_1\bar{\gamma}_2)A + (1-\gamma_2\bar{\gamma}_2)A'] - \rho_2[(1-\gamma_1\bar{\gamma}_1)A + (1-\bar{\gamma}_1\gamma_2)A']}{\bar{\gamma}_1 - \bar{\gamma}_2},$$

$$+ e^{-2i\theta}\left\{\rho_1\bar{A} + \rho_2\bar{A}' - \frac{\delta_1[(1-\bar{\gamma}_1\gamma_2)\bar{A} + (1-\gamma_2\bar{\gamma}_2)\bar{A}'] - \delta_2[(1-\gamma_1\bar{\gamma}_1)\bar{A} + (1-\gamma_1\bar{\gamma}_2)\bar{A}]}{\gamma_1 - \gamma_2}\right\}. \tag{5.24}$$

Upon substitution of expressions (5.23) into (5.24) the following result is obtained after some manipulation:

$$\frac{u_r + iu_\theta}{a} = -\frac{\ell_{22}}{(1-\gamma_1)^2(1-\gamma_2)^2}\Big\{[1+(\gamma_1+\gamma_2)^2 - \gamma_1^2\gamma_2^2](\sigma_1+\sigma_2) + 2(\gamma_1+\gamma_2)(\sigma_1-\sigma_2)$$

$$-2i(1-\gamma_1\gamma_2)(\gamma_1+\gamma_2)\tau_{12}$$

$$+ e^{-2i\theta}[2(\gamma_1+\gamma_2)(\sigma_1+\sigma_2) + 2(1+\gamma_1\gamma_2)(\sigma_1-\sigma_2)$$

$$+ 4i(1-\gamma_1\gamma_2)\tau_{12}]\Big\}. \tag{5.25}$$

Now from the definition of γ_1 and γ_2 in (5.19), they are either both real, or complex conjugates. The separation of (5.25) into real and imaginary parts then follows easily, and we have for the radial displacement due to plane-strain:

$$\frac{u_r}{a} = A' + B' \cos 2\theta + C' \sin 2\theta \ , \tag{5.26}$$

where

$$A' = - \frac{\ell_{22}\left\{[1 + (\gamma_1 + \gamma_2)^2 - \gamma_1^2 \gamma_2^2](\sigma_1 + \sigma_2) + 2(\gamma_1 + \gamma_2)(\sigma_1 - \sigma_2)\right\}}{(1 - \gamma_1)^2 (1 - \gamma_2)^2} \ ,$$

$$B' = - \frac{2\ell_{22}[(\gamma_1 + \gamma_2)(\sigma_1 + \sigma_2) + (1 + \gamma_1 \gamma_2)(\sigma_1 - \sigma_2)]}{(1 - \gamma_1)^2 (1 - \gamma_2)^2} \ , \tag{5.27}$$

$$C' = - \frac{4\ell_{22}(1 - \gamma_1 \gamma_2)\tau_{12}}{(1 - \gamma_1)^2 (1 - \gamma_2)^2} \ .$$

Briefer expressions may be obtained (as in the stress solutions of I) by using the constants

$$k_1 = \alpha_1 \alpha_2 \ , \quad k_2 = \tfrac{1}{2}(\alpha_1^2 + \alpha_2^2) \ , \quad k_3 = \sqrt{2(k_1 + k_2)} = \alpha_1 + \alpha_2 \ . \tag{5.28}$$

Then

$$\begin{aligned} A' &= - \tfrac{1}{4}\ell_{22}\left\{[(k_1 - 1)^2 + k_3(k_1 + 1)](\sigma_1 + \sigma_2) + (k_1 - 1)(k_1 + k_3 + 1)(\sigma_1 - \sigma_2)\right\} \ , \\ B' &= - \tfrac{1}{4}\ell_{22}(k_1 + k_3 + 1)\left\{(k_1 - 1)(\sigma_1 + \sigma_2) + (k_1 + 1)(\sigma_1 - \sigma_2)\right\} \ , \\ C' &= - \tfrac{1}{2}\ell_{22}k_3(k_1 + k_3 + 1)\tau_{12} \ . \end{aligned} \tag{5.29}$$

By substitution of values equal in magnitude but opposite in sign to the regional stress components in the third of (5.14) we find the constant axial strain,

$$\varepsilon_z = -(b_{12}\sigma_1 + b_{13}\sigma_2 + b_{11}\sigma_3) , \tag{5.30}$$

and calculate the additional radial displacement caused by this departure from plane-strain. Setting $\sigma_x = \sigma_y = 0$ in (5.14) we find the relations between this and the additional strain components ε_x^o and ε_y^o:

$$\varepsilon_x^o = b_{12}\sigma_x^o , \qquad \varepsilon_y^o = b_{13}\sigma_z^o ,$$

$$\varepsilon_z = b_{11}\sigma_z^o . \tag{5.31}$$

However, since we are interested in the radial strain in a direction at an angle θ to the x-axis we use the transformation

$$\varepsilon_r = \varepsilon_x \cos^2\theta + \varepsilon_y \sin^2\theta = \tfrac{1}{2}[\,\varepsilon_x + \varepsilon_y + (\varepsilon_x - \varepsilon_y)\cos 2\theta\,] ,$$

so that, from (5.31),

$$\varepsilon_r^o = \frac{\varepsilon_z}{2b_{11}}\,[\,b_{12} + b_{13} + (b_{12} - b_{13})\cos 2\theta\,] .$$

At the hole this corresponds to the comparative radial displacement u_r^o / a, and so, using (5.30),

$$\frac{u_r^o}{a} = -\frac{1}{2b_{11}}(b_{12}\sigma_1 + b_{13}\sigma_2 + b_{11}\sigma_3)[\,b_{12} + b_{13} + (b_{12} - b_{13})\cos 2\theta\,] . \tag{5.32}$$

By means of I(1.3) this expression can be put into the form

$$\frac{u_r^o}{a} = -\frac{1}{4E_1}\Big\{(\nu_1+\nu_2)^2(\sigma_1+\sigma_2) + (\nu_1^2-\nu_2^2)(\sigma_1-\sigma_2) - 2(\nu_1+\nu_2)\,\sigma_3$$

$$+ [\,(\nu_1^2-\nu_2^2)(\sigma_1+\sigma_2) + (\nu_1-\nu_2)^2(\sigma_1-\sigma_2) - 2(\nu_1-\nu_2)\,\sigma_3]\cos 2\theta\Big\} \tag{5.33}$$

which can then be combined with the plane-strain solution given by (5.26) and (5.29) to obtain the complete solution:

$$\frac{u_r}{a} = A_{90} + B_{90}\cos 2\theta + C_{90}\sin 2\theta \,, \tag{5.34}$$

where

$$A_{90} = -\tfrac{1}{4}\left\{\ell_{22}(k_1-1)^2 + k_3(k_1+1)] + \frac{(\nu_1+\nu_2)^2}{E_1}\right\}(\sigma_1+\sigma_2)$$

$$-\tfrac{1}{4}\left\{\ell_{22}(k_1-1)(k_1+k_3+1) + \frac{\nu_1^2-\nu_2^2}{E_1}(\sigma_1-\sigma_2)\right\} + \frac{\nu_1+\nu_2}{2E_1}\,\sigma_3 \,,$$

$$B_{90} = -\tfrac{1}{4}\left\{\ell_{22}(k_1-1)(k_1+k_3+1) + \frac{\nu_1^2-\nu_2^2}{E_1}\right\}(\sigma_1+\sigma_2) \tag{5.35}$$

$$-\tfrac{1}{4}\left\{\ell_{22}(k_1+1)(k_1+k_3+1) + \frac{(\nu_1-\nu_2)^2}{E_1}\right\}(\sigma_1-\sigma_2) + \frac{\nu_1-\nu_2}{2E_1}\,\sigma_3 \,,$$

$$C_{90} = -\tfrac{1}{2}\ell_{22}k_3(k_1+k_3+1)\,\tau_{12}.$$

The constant ℓ_{22} can be expressed in terms of E_1, ν_1 and k_1 thus:

$$\ell_{22} = \frac{1-\nu_1^2}{E_1k_1^2} \,. \tag{5.36}$$

5.3 Isotropy

The result for an isotropic material may be obtained by letting $E_1 = E_2 = E$, $\nu_1 = \nu_2 = \nu$, $k_1 = 1$ and $k_3 = 2$ in (5.13) or (5.35). These expressions then reduce to

$$A = -\frac{1}{E}(\sigma_1 + \sigma_2 - \nu\sigma_3) ,$$

$$B = -\frac{2}{E}(1 - \nu^2)(\sigma_1 - \sigma_2) , \tag{5.37}$$

$$C = -\frac{4}{E}(1 - \nu^2)\ \tau_{12} ,$$

which, of course, are the coefficients in the expression

$$\frac{u_r}{a} = A + B\ \cos\theta + C\ \sin\theta . \tag{5.38}$$

6. Application of the radial displacement formulae

6.1 Development of procedure

In each of the expressions for the radial displacement, (4.17), (5.11), (5.34), the result depends on only three parameters which are constant for any particular hole. No matter how many measurements of u_r are made at various directions θ in that hole, it is not possible to gain any more information than that of the values of the parameters A_ϕ, B_ϕ, C_ϕ. However, in order to determine the six components of regional stress, it is necessary to determine the values of six parameters. To do so, measurements must be made in a second hole with a different orientation.*

In each hole, a minimum of three diametral strain measurements is adequate but more may be made to improve accuracy (see Panek, 1966, for procedure in the isotropic case). Denoting the three measurements by suffixes 1,2,3 we have

$$\frac{u_{r,1}}{a} = A + B\cos 2\theta_1 + C\sin 2\theta_1 ,$$

$$\frac{u_{r,2}}{a} = A + B\cos 2\theta_2 + C\sin 2\theta_2 , \tag{6.1}$$

$$\frac{u_{r,3}}{a} = A + B\cos 2\theta_3 + C\sin 2\theta_3 .$$

* In some circumstances measurements in a third hole may be necessary: see note at end of Section 6.2.

The determinant of the coefficients of A, B, C in (6.1) is $4 \sin(\theta_1 - \theta_2) \sin(\theta_2 - \theta_3) \sin(\theta_3 - \theta_1)$ showing that these parameters may always be determined.

When measurements in two holes, with orientations ϕ_1 and ϕ_2 (say), have been made, six parameters, A_1, B_1, C_1, A_2, B_2, C_2, are available for determination of the regional stress components. For definiteness, we suppose that $0 \leq \phi_1 \leq \phi_2 \leq \pi/2$, and so indicate the place of the exceptional values of ϕ in the scheme.

It is important to remember that the regional stress components in the form σ_1, σ_2, σ_3, τ_{12}, τ_{23}, τ_{31} are referred to axes fixed on the axis of the hole and hence will change value as the orientation of the hole is changed. Before proceeding to solve the six equations for the regional stress components the formulae for A_ϕ, B_ϕ, C_ϕ must be expressed in terms of components referred to a single system of coordinates. There are several reasonable systems, but it seems preferable to choose one in which one axis is an axis of elastic symmetry. We denote the chosen system by (x_1^R, x_2^R, x_3^R) and the components of regional stress R_{ij}. Then we choose Ox_3^R to be an axis of elastic symmetry, so that it is identical with Ox_3'' (Fig. 3). We could choose Ox_1^R to be a geographical direction, or to be identical with the x_1-direction defined for either of the holes (but not by both unless both holes lie in a plane through an axis of elastic symmetry) (Figs. 1 and 2).

Let ψ_1 be the angle made by the x_1-direction of the first hole to Ox_1^R, and ψ_2 the angle made by that of the second hole. The direction cosines, $d_{ij}^m = \cos(x_i^m O x_j^R)$ $(m = 1, 2)$, of the x_i-directions defined for the first hole $(m = 1)$ and second hole $(m = 2)$ with respect to the (x_j^R) coordinates are

$$d_{11}^m = \cos\psi_m , \qquad d_{12}^m = \sin\psi_m , \qquad d_{13}^m = 0 ,$$

$$d_{21}^m = -\sin\psi_m \cos\phi_m , \; d_{22}^m = \cos\psi_m \cos\phi_m , \; d_{23}^m = -\sin\phi_m , \tag{6.2}$$

$$d_{31}^m = -\sin\psi_m \sin\phi_m , \; d_{32}^m = \cos\psi_m \sin\phi_m , \; d_{33}^m = \cos\phi_m .$$

If we choose Ox_1^R to be identical with Ox_1 for $m = 1$ (say), then $\psi_1 = 0$, with consequent simplification in that case.

If we denote by σ^m_{ij} $(m = 1,2)$ the regional stress components referred to the x^1_i - coordinates and x^2_i - coordinates respectively, then they may be expressed in terms of the R_{ij} (the regional stress components referred to the x^R_i - coordinates) by means of the transformations

$$\sigma^m_{ij} = d_{ik} d_{j\ell} R_{k\ell} \qquad \text{(summed over } k, \ell \text{)}. \qquad (6.3)$$

Written out in the original notation for the regional stress components this gives the expressions

$$\sigma^m_1 = R_{11}\cos^2\psi_m + R_{22}\sin^2\psi_m + 2R_{12}\sin\psi_m\cos\psi_m ,$$

$$\sigma^m_2 = R_{11}\sin^2\psi_m\cos^2\phi_m + R_{22}\cos^2\psi_m\cos^2\phi_m + R_{33}\sin^2\phi_m - 2R_{12}\sin\psi_m\cos\psi_m\cos^2\phi_m$$
$$-2R_{23}\cos\psi_m\sin\phi_m\cos\phi_m + 2R_{13}\sin\psi_m\sin\phi_m\cos\phi_m ,$$

$$\sigma^m_3 = R_{11}\sin^2\psi_m\sin^2\phi_m + R_{22}\cos^2\psi_m\sin^2\phi_m + R_{33}\cos^2\phi_m$$
$$-2R_{12}\sin\psi_m\cos\psi_m\sin^2\phi_m + 2R_{23}\cos\psi_m\sin\phi_m\cos\phi_m - 2R_{13}\sin\psi_m\sin\phi_m\cos\phi_m ,$$

$$\tau^m_{12} = -R_{11}\sin\psi_m\cos\psi_m\cos\phi_m + R_{22}\sin\psi_m\cos\psi_m\cos\phi_m + R_{12}\cos\phi_m(\cos^2\psi_m - \sin^2\psi_m)$$
$$-R_{23}\sin\psi_m\sin\phi_m - R_{13}\cos\psi_m\sin\phi_m ,$$

$$(6.4)$$

$$\tau^m_{23} = R_{11}\sin^2\psi_m\sin\phi_m + R_{22}\cos^2\psi_m\sin\phi_m\cos\phi_m - R_{33}\sin\phi_m\cos\phi_m$$
$$-2R_{12}\sin\psi_m\cos\psi_m\sin\phi_m\cos\phi_m + R_{23}\cos\psi_m(\cos^2\phi_m - \sin^2\phi_m)$$
$$+R_{13}\sin\psi_m(\sin^2\phi_m - \cos^2\phi_m),$$

$$\tau^m_{13} = -R_{11}\sin\psi_m\cos\psi_m\sin\phi_m + R_{22}\sin\psi_m\cos\psi_m\sin\phi_m + R_{12}\sin\phi_m(\cos^2\psi_m - \sin^2\psi_m)$$
$$+R_{23}\sin\psi_m\cos\phi_m + R_{13}\cos\psi_m\cos\phi_m .$$

The exceptional cases $\phi_1 = 0$, $\phi_2 = \pi/2$ bring simplifications, particularly if ψ_1 or ψ_2 are chosen conveniently.

(i) $\underline{\phi_1 = 0;\ \psi_1 = 0}$. In this case the x_1^1 - direction is chosen arbitrarily in the plane of symmetry and the x_1-direction is chosen to be identical with it. Then, from (6.4)

$$\left.\begin{aligned} \sigma_1^1 &= R_{11}, \quad \sigma_2^1 = R_{22}, \quad \sigma_3^1 = R_{33}, \\ \tau_{12}^1 &= R_{12}, \quad \tau_{23}^1 = R_{23}, \quad \tau_{13}^1 = R_{13}, \end{aligned}\right\} \tag{6.5}$$

and the formulae (5.13) become

$$\left.\begin{aligned} A_o &= -\frac{1}{E_1}(R_{11} + R_{22} - \nu_2 R_{33}), \\ B_o &= -\frac{2}{E_1}(1 - \nu_2^2)(R_{11} - R_{22}), \\ C_o &= -\frac{4}{E_1}(1 - \nu_2^2) R_{12}. \end{aligned}\right\} \tag{6.6}$$

(ii) $\underline{\phi_2 = \pi/2;\ \psi_2 = 0}$. This choice of ψ_2 means that the

x_1^R - direction is in the plane of cross-section of the hole, and the x_2^R - direction coincides with the axis of the hole. Since the x_1^1 - axis for $\phi_1 = 0$ in (i) above can have any direction in the plane of elastic symmetry, the choice of $\psi = 0$ in each case does not prevent the selection of a common x_i^R - coordinate system. From (6.4), we have then

$$\left.\begin{aligned} \sigma_1^2 &= R_{11}, \quad \sigma_2^2 = R_{33}, \quad \sigma_3^2 = R_{22}, \\ \tau_{12}^2 &= -R_{13}, \quad \tau_{23}^2 = -R_{23}, \quad \tau_{13}^2 = R_{12}, \end{aligned}\right\} \tag{6.7}$$

and the formulae (5.35) become, using (5.36),

$$A_{90} = -\frac{1}{4E_1}\left\{\frac{1-\nu_1^2}{k_1^2}\left[(k_1-1)^2 + k_3(k_1-1)\right] + (\nu_1+\nu_2)^2\right\}(R_{11}+R_{33})$$

$$-\frac{1}{4E_1}\left\{\frac{1-\nu_1^2}{k_1^2}(k_1-1)(k_1+k_3+1)+\nu_1^2-\nu_2^2\right\}(R_{11}-R_{33})+\frac{\nu_1+\nu_2}{2E_1}R_{22}\ ,$$

$$B_{90} = -\frac{1}{4E_1}\left\{\frac{1-\nu_1^2}{k_1^2}(k_1-1)(k_1+k_3+1)+\nu_1^2-\nu_2^2\right\}(R_{11}+R_{33}) \tag{6.8}$$

$$-\frac{1}{4E_1}\left\{\frac{1-\nu_1^2}{k_1^2}(k_1+1)(k_1+k_3+1)+(\nu_1-\nu_2)^2\right\}(R_{11}-R_{33})-\frac{\nu_1-\nu_2}{2E_1}R_{22}\ ,$$

$$C_{90} = \frac{(1-\nu_1^2)\,k_3\,(k_1+k_3+1)}{2E_1k_1^2}R_{13}\ .$$

Unfortunately, we cannot make appropriate substitutions (as we have for $\phi = 0,\ \pi/2$) in the expressions A_ϕ, B_ϕ, C_ϕ (4.18), for $0 < \phi < \pi/2$, to obtain formula with explicit dependence upon the R_{ij}, because their dependence on the regional stress components arises in a complicated manner through the $a_{\nu o}$ and A_ν (and, more simply, through ϵ_z). The dependence of the $a_{\nu o}$ on σ_1, σ_2, τ_{12}, τ_{13}, τ_{23} can be found only by solving equations I(4.2), and that is not practicable until numerical values can be assigned to the coefficients (calculated from the elastic properties of a particular material with the appropriate ϕ in each case). The A_ν can then be calculated from I(4.12). The component σ_3 is introduced through ϵ_z, which is given in terms of stress components by (3.2). In a practical case, $a_{\nu o}$, A_ν and ϵ_z would then be expressed in terms of the R_{ij}, using the appropriate values for the angles ϕ_m and ψ_m for each hole in (6.4).

The elastic parameters G_ν and H_ν also depend upon ϕ, and the values appropriate to ϕ_1 and ϕ_2 must be calculated. All the equations are linear in stress components and when all the substitutions have been made we obtain expressions A_1, B_1, C_1 for $\phi = \phi_1$, $\psi = \psi_1$, linear and homogeneous in the stress components R_{ij}, and a similar set A_2, B_2, C_2 for $\phi = \phi_2$, $\psi = \psi_2$. If we denote by A_1^e etc. the experimental values determined through equations of the form of (6.1), we can write the result as six equations thus:

$$
\begin{aligned}
a_1^1 R_{11} + a_2^1 R_{22} + a_3^1 R_{33} + a_4^1 R_{23} + a_5^1 R_{31} + a_6^1 R_{12} &= A_1^e \\
b_1^1 R_{11} + b_2^1 R_{22} + b_3^1 R_{33} + b_4^1 R_{23} + b_5^1 R_{31} + b_6^1 R_{12} &= B_1^e \\
c_1^1 R_{11} + c_2^1 R_{22} + c_3^1 R_{33} + c_4^1 R_{23} + c_5^1 R_{31} + c_6^1 R_{12} &= C_1^e \\
a_1^2 R_{11} + a_2^2 R_{22} + a_3^2 R_{33} + a_4^2 R_{23} + a_5^2 R_{31} + a_6^2 R_{12} &= A_2^e \\
b_1^2 R_{11} + b_2^2 R_{22} + b_3^2 R_{33} + b_4^2 R_{23} + b_5^2 R_{31} + b_6^2 R_{12} &= B_2^e \\
c_1^2 R_{11} + c_2^2 R_{22} + c_3^2 R_{33} + c_4^2 R_{33} + c_5^2 R_{31} + c_6^2 R_{12} &= C_2^e \ .
\end{aligned}
\tag{6.9}
$$

So long as the matrix of coefficients a_1^1 etc. in (6.9) is non-singular, the equations may be solved for the R_{ij}. For $\phi_1 = 0$, $\psi_1 = 0$ expressions (6.6) are substituted for the first three expressions in (6.9). If $\phi_2 = {}^{\pi}/_2$, $\psi_2 = 0$ expressions (6.8) are substituted for the second three of (6.9). A complete solution cannot be obtained by having $\phi_1 = 0$ and $\phi_2 = {}^{\pi}/_2$ simultaneously, since R_{23} occurs in neither (6.6) nor (6.8) and the matrix of coefficients of the six equations is singular: by omitting an equation a solution could be obtained for the other five components. It does not seem practicable to determine whether there are any other conditions for which the matrix is non-singular. However, the possibility of this occurring must not be overlooked and, indeed, steps should be taken to ensure that the matrix is "well-conditioned" in practical applications. This could be done by evaluating its determinant at a selection of possible relative orientations of the holes, and rejecting those giving the smaller values. One would be especially suspicious of holes at right-angles to each other.

6.2 Summary of method of application

(1) The constants k_{rs} are evaluated in terms of the b_{rs} (assumed known) for a pair of values ϕ_1 and ϕ_2, through I(1.6). The constants ℓ_{rs}, K_s, λ_ν, μ_ν, γ_ν, δ_ν, $\gamma_{\nu\nu}$, $\delta_{\nu\nu}$, $\mu_{\nu\nu}$, $L_{1\nu}$, $L_{2\nu}$, G_ν, H_ν ($\nu = 1,2,3$) are then determined for each of ϕ_1 and ϕ_2 by means of I(2.6'), I(2.17), I(2.18), I(2.20), I(4.6), I(4.8), I(4.9), (2.8) and (4.16).

(2) The equations I(4.2) are solved for a_{10}, a_{20}, $\bar{a}_{20}$, a_{30}, $\bar{a}_{30}$ in terms of σ_1, σ_2, τ_{12}, τ_{13}, τ_{23}, using the coefficients appropriate to each of the values ϕ_1 and ϕ_2 in turn. These are substituted in I(4.12) to obtain the values of $A_\nu (\nu = 1,2,3)$ (using the appropriate coefficients for each of ϕ_1 and ϕ_2) in terms of σ_1, σ_2, τ_{12}, τ_{13}, τ_{23}. The values of the k_{rs} for ϕ_1 and ϕ_2 are used in turn in (3.2) to obtain the values of ϵ_z in each case, in terms of σ_1, σ_2, σ_3, τ_{23}.

(3) The orientation of the x_1^R, x_2^R- axes in the plane of elastic symmetry is selected. The x_1^1 - direction is defined (Figs. 1 and 2) by the intersection of a cross-section plane of the first hole with a plane of elastic symmetry, and ψ_1 is defined as the angle it makes to the x_1^R- direction: ψ_2 is defined similarly.

(4) The values of the $a_{\nu o}$, $\bar{a}_{\nu o}$, A_ν and ϵ_z for each of ϕ_1 and ϕ_2 are expressed in terms of the R_{ij} by means of (6.4), using the ψ_m appropriate for each hole.

(5) The resulting expressions for $a_{\nu o}$, $\bar{a}_{\nu o}$, A_ν, $\bar{A}_\nu$ and ϵ_z for $\phi = \phi_1$, $\psi = \psi_1$ are substituted in (4.18) with coefficients appropriate to ϕ_1 and form the first three expressions in (6.9). A similar process for $\phi = \phi_2$, $\psi = \psi_2$ gives the second three expressions. If $\phi_1 = 0$, the expressions (6.6) can form the first three of (6.7); if $\phi_2 = \pi/2$, (6.8) can form the second three of (6.9), so long as the x_i^R - coordinates are appropriately selected.

(6) The determinant of the coefficients of the R_{ij} in the resulting set of expressions is evaluated to ensure that it is not zero or very small.

(7) If this test gives a satisfactory result, stress-relief tests are made in boreholes at the selected angles. Diametral measurements are made in three directions to obtain a set of equations of the form (6.1) for each borehole. From the first hole we calculate dimensionless parameters A_1^e, B_1^e, C_1^e, and from the second parameters A_2^e, B_2^e, C_2^e. These form the right-hand sides of equations (6.9) which can now be solved for the regional stress components R_{ij}.

Note If the ground is only weakly anisotropic it is probable that the equations (6.9) may not be well-conditioned since, as shown by Gray and Toews (1967), and also by Bonnechere and

Fairhurst (1968) with reference to the "doorstopper" method, the six equations obtained from any pair of holes are not independent when the ground is isotropic and measurements in three holes are necessary. The arguments of the above authors do not apply to ground which is only transversely isotropic, although instances of dependence between the six equations (6.9) other than that noted in Section 6.1 ($\phi_1 = 0$, $\phi_2 = \pi/2$) may exist. If the matrix of coefficients of (6.9) is zero, or close to zero (either because of weakness of the anisotropy or because of limitations in choice of hole-direction), it will be necessary to use a third hole ($\phi = \phi_3$, $\psi = \psi_3$) in addition, and the results may be processed by methods discussed by Panek (1966) and Gray and Toews (1967).

7. Strain in the wall of a stress-relieved borehole

Development of very small strain-gauges makes it feasible to measure the strain in the wall of a borehole as the stress is relieved. This technique could be used to augment information obtained from the older method, and possibly to allow determination of the regional stress tensor from measurements in a single borehole.

The components of strain in the wall of the hole, r = a, in the (r, θ, z) coordinate system are ϵ_θ, ϵ_z, $\gamma_{\theta z}$. Of these ϵ_z is constant and is given by (3.2), while

$$\epsilon_\theta = \epsilon_x \sin^2\theta - 2\gamma_{xy}\sin\theta\cos\theta + \epsilon_y \cos^2\theta , \qquad (7.1)$$

$$\gamma_{\theta z} = \gamma_{yz}\cos\theta - \gamma_{xz}\sin\theta \qquad (7.2)$$

7.1 General case, $o < \phi < \pi/2$

In this case ϵ_x, ϵ_y, γ_{xy}, γ_{yz}, γ_{xz} are given by equations (2.3) to (2.8) and ϵ_z by (3.2), with the $W_\nu(z) = -[W_\nu(z_\nu)]_1$ as given by (4.9). From (7.1), (2.3), (2.4) and (2.5), we find that

$$\epsilon_\theta = \sum_{\nu=1}^{3} [M_\nu W_\nu(z_\nu) + \bar{M}_\nu \bar{W}_\nu(z_\nu)] + (K_1 \sin^2\theta + K_2 \cos^2\theta) \epsilon_z , \qquad (7.3)$$

where

$$M_\nu = L_{1\nu}\sin^2\theta + L_{2\nu}\cos^2\theta - L_{6\nu}\sin\theta\cos\theta , \qquad (7.4)$$

$L_{1\nu}$, $L_{2\nu}$ and $L_{6\nu}$ being given by (2.8). For the boundary of the hole, $r = a$, we use the value of $[\, W_\nu(z_\nu)\,]_1$, given by I(4.13), so that then

$$W_\nu(z_\nu) = - \frac{(a_{\nu o}\gamma_\nu - A_\nu)\sin\theta - i\,(a_{\nu o}\gamma_\nu + A_\nu)\cos\theta}{\sin\theta - \lambda_\nu\cos\theta} \qquad (7.5)$$

From (7.2), (2.6) and (2.7), we find that

$$\gamma_{\theta z} = \sum_{\nu=1}^{3} [\, N_\nu W_\nu(z_\nu) + \bar{N}_\nu \bar{W}_\nu(\bar{z}_\nu)\,] + K_4 \cos\theta\, \epsilon_z\, , \qquad (7.6)$$

where

$$N_\nu = \tfrac{1}{2}\,(L_{4\nu}\cos\theta - L_{5\nu}\sin\theta)\, , \qquad (7.7)$$

$L_{4\nu}$ and $L_{5\nu}$ being given by (2.8), and $W_\nu(z_\nu)$ by (7.5), for $r = a$. In both (7.3) and (7.6), the value of ϵ_z is given by (3.2):

$$\epsilon_z = -\,(k_{13}\sigma_1 + k_{23}\sigma_2 + k_{33}\sigma_3 + k_{34}\tau_{23})\, . \qquad (7.8)$$

Equations (7.3) to (7.8) give the distribution of strain in the wall of the hole as a function of the elastic constants, hole orientation and regional stress tensor. The dependence of $a_{\nu o}$ and A_ν on the components of regional stress must, in practice, be calculated by the methods indicated in Section 6.

7.2 Case $\phi = 0$.

In the case $\phi = 0,\ \pi/2$, the plane deformations and the axial shear deformations are "uncoupled" and so ϵ_θ and $\gamma_{\theta z}$ may be calculated as separate problems.

Transformation of the plane-strain relations (5.1) to the (r, θ, z) system of coordinates gives the relation

$$\epsilon_\theta = b_{11}\sigma_\theta + b_{12}\sigma_r + b_{13}\sigma_z$$

but since $\sigma_r = 0$ on $r = a$, we have

$$\epsilon_\theta = b_{11}\sigma_\theta + b_{13}\sigma_z\, , \text{ on } r = a \qquad (7.9)$$

where b_{11} and b_{13} are given by I(1.3).

The values of σ_θ and σ_z may be obtained from I(5.5) and I(5.7) with the signs changed (since here the regional stress is being removed):

$$\sigma_\theta = -(\sigma_1 + \sigma_2) + 2(\sigma_1 - \sigma_2)\cos 2\theta + 4\tau_{12}\sin 2\theta , \qquad (7.10)$$

$$\sigma_z = -\sigma_3 - \frac{b_{13}}{b_{33}}\left[2(\sigma_1 - \sigma_2)\cos 2\theta + 4\tau_{12}\sin 2\theta \right] . \qquad (7.11)$$

Substitution of (7.10) and (7.11) into (7.9) give the result for $r = a$,

$$\epsilon_\theta = -b_{11}(\sigma_1 + \sigma_2) - b_{13}\sigma_3 + 2\left(b_{11} - \frac{b_{13}^2}{b_{33}}\right)\left[(\sigma_1 - \sigma_2)\cos 2\theta + 2\tau_{12}\sin 2\theta\right] \qquad (7.12)$$

From I(1.3) this may be written

$$E_1\epsilon_\theta = -(\sigma_1 + \sigma_2) + \nu_2\sigma_3 + 2\left(1 - \frac{E_2}{E_1}\nu_2^2\right)\left[(\sigma_1 - \sigma_2)\cos 2\theta + 2\tau_{12}\sin 2\theta\right] . \qquad (7.13)$$

By changing the sign of I(5.19) (because the regional stress is removed) we have the component of shear stress in the wall of the hole:

$$\tau_{\theta z} = -2(\tau_{23}\cos\theta - \tau_{13}\sin\theta) . \qquad (7.14)$$

(From (5.1) (suitably transformed) we have the relation

$$2\gamma_{\theta z} = b_{44}\tau_{\theta z} ,$$

and so, from (7.14)

$$\gamma_{\theta z} = b_{44}(\tau_{13}\sin\theta - \tau_{23}\cos\theta) , \qquad (7.15)$$

or, using I(1.3),

$$M\gamma_{\theta z} = \tau_{13}\sin\theta - \tau_{23}\cos\theta. \qquad (7.16)$$

The remaining component, ϵ_z is given by equation (5.9), which, by use of I(1.3) can be written:

$$E_1\epsilon_z = \nu_2(\sigma_1 + \sigma_2) - \frac{E_1}{E_2}\sigma_3 . \tag{7.17}$$

Equations (7.13), (7.16) and (7.17) give the components of strain in the wall of the hole directly in terms of the elastic constants and the regional stress components referred to axes fixed on the hole.

7.3 Case $\phi = \pi/2$

A slightly different procedure is used for $\phi = \pi/2$ in order to avoid the labour of transforming the equations (5.14), which apply to this orientation of the hole.

The stress components σ_x, σ_y, τ_{xy} are derived from components referred to the (r, θ, z) coordinates by the transformation

$$\left.\begin{aligned} \sigma_x &= \sigma_r\cos^2\theta + \sigma_\theta\sin^2\theta - 2\tau_{r\theta}\sin\theta\cos\theta , \\ \sigma_y &= \sigma_r\sin^2\theta + \sigma_\theta\cos^2\theta + 2\tau_{r\theta}\sin\theta\cos\theta , \\ \tau_{xy} &= (\sigma_r - \sigma_\theta)\sin\theta\cos\theta - \tau_{r\theta}(\cos^2\theta - \sin^2\theta). \end{aligned}\right\} \tag{7.18}$$

In the present case, for r = a, $\sigma_r = \tau_{r\theta} = 0$, and σ_θ is given by I(5.40) and I(5.41), and σ_z by I(5.46) both with the signs of the regional stress components changed (since they are being removed, not applied).

From (5.14)

$$\left.\begin{aligned} \epsilon_x &= b_{11}\sigma_x + b_{13}\sigma_y + b_{12}\sigma_z , \\ \epsilon_y &= b_{13}\sigma_x + b_{33}\sigma_y + b_{13}\sigma_z , \\ 2\gamma_{xy} &= b_{44}\tau_{xy}. \end{aligned}\right\} \tag{7.19}$$

Substitution of (7.18) in (7.19) and use of (7.1) gives the result

$$\epsilon_\theta = \sigma_\theta(b_{11}\sin^4\theta + 2b_{13}\sin^2\theta\cos^2\theta + b_{33}\cos^4\theta + b_{44}\sin^2\theta\cos^2\theta)$$

$$+ \sigma_z(b_{12}\sin^2\theta + b_{13}\cos^2\theta). \qquad (7.20)$$

From I(1.3) and I(5.46), this can be written

$$E_1\epsilon_\theta = \sigma_\theta[\sin^4\theta - 2\nu_2\sin^2\theta\cos^2\theta + \frac{E_1}{E_2}\cos^4\theta + \frac{E_1}{M}\sin^2\theta\cos^2\theta -$$

$$- (\nu_1\sin^2\theta + \nu_2\cos^2\theta)^2\,] + (\nu_1\sin^2\theta + \nu_2\cos^2\theta)(\sigma_3 - \nu_1\sigma_1 - \nu_2\sigma_2),$$

$$(7.21)$$

where, from I(5.40) and I(5.41)

$$H\cdot\sigma_\theta = -(\sigma_1+\sigma_2)[\,k_3(k_1+1)+(k_1-1)^2 - (k_1-k_3+1)(k_1-1)\cos 2\theta\,]$$

$$-(\sigma_1-\sigma_2)(k_1+k_3+1)[\,k_1-1-(k_1+1)\cos 2\theta\,] + 2\tau_{12}k_3(k_1+k_3+1)\sin 2\theta,$$

$$(7.22)$$

$$H = (k_1^2 + 2k_2 + 1) - 2(k_1^2-1)\cos 2\theta + (k_1^2-2k_2+1)\cos^2 2\theta \qquad (7.23)$$

To calculate the component $\gamma_{\theta z}$ we use (7.2) in conjunction with the relations from (5.14):

$$\left.\begin{aligned} \gamma_{yz} &= \tfrac{1}{2}b_{44}\tau_{yz}, \\ \gamma_{xz} &= (b_{11} - b_{12})\,\tau_{xz}, \end{aligned}\right\} \qquad (7.24)$$

and the expressions I(5.66) for τ_{yz} and τ_{xz}, with signs changed. The result is

$$\gamma_{\theta z} = \frac{(1 + t)(\tau_{13}\sin\theta - t\tau_{23}\cos\theta)}{\sin^2\theta + t^2\cos^2\theta}\left[\tfrac{1}{2}b_{44}\cos^2\theta + (b_{11}-b_{12})\sin^2\theta\right], \qquad (7.25)$$

where, from I(5.68)

$$t^2 = \frac{b_{44}}{2(b_{11} - b_{12})} = \frac{E_1}{2M(1 + \nu_1)} . \qquad (7.26)$$

Using I(1.3) and (7.26), this may be written

$$E_1\gamma_{\theta z} = (1 + \nu_1)(1 + t)(\tau_{13}\sin\theta - t\tau_{23}\cos\theta) . \qquad (7.27)$$

The axial strain is given simply by I(5.3), which may be written, by means of I(1.3):

$$E_1\epsilon_z = -\sigma_3 + \nu_1\sigma_1 + \nu_2\sigma_2 . \qquad (7.28)$$

Equations (7.21), (7.22), (7.23), (7.27) and (7.28) give the components of strain in the wall of the hole (for $\phi = \pi/2$) directly in terms of the elastic constants referred to axes fixed on the hole.

7.3 Isotropic Material

The results for an isotropic material may be obtained simply by putting $E_1 = E_2 = E$, $\nu_1 = \nu_2 = \nu$, $M = \mu = \frac{1}{2}E/(1 + \nu)$, in (7.13), (7.16) and (7.17). The results are

$$E\epsilon_\theta = -(\sigma_1+\sigma_2) + \nu\sigma_3 + 2(1-\nu^2)[(\sigma_1-\sigma_2)\cos 2\theta + 2\tau_{12}\sin 2\theta], \qquad (7.29)$$

$$E\gamma_{\theta z} = 2(1 + \nu)(\tau_{13}\sin\theta - \tau_{23}\cos\theta), \qquad (7.30)$$

$$E\epsilon_z = \nu(\sigma_1 + \sigma_2) - \sigma_3 . \qquad (7.31)$$

7.4 Measurement

In order to compare the results of this section with measured values we require the normal components of the strains calculated above resolved in any direction tangential to the wall of the hole. Let ϵ_ω be the normal component of strain in the wall in a direction at an angle ω to the positive θ-direction at any point then its value in terms of ϵ_θ, ϵ_z, $\gamma_{\theta z}$ at that point is given by the transformation

$$\epsilon_\omega = \epsilon_\theta \cos^2\omega + \epsilon_z \sin^2\omega + 2\gamma_{\theta z} \sin\omega\cos\omega . \tag{7.32}$$

If a triple strain-gauge rosette were used, with gauges at equal angular intervals, then the three angles $0, \pi/3, 2\pi/3$, or $\pi/6, \pi/2, 5\pi/6$ would be obvious choices for ω.

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13. ABSTRACT

This report is part of a continuing study into Methods of Determining In-Situ Stresses at Great Depths. Results of the first part of this study were reported in Technical Report No. 1-68, ti 'e as underlined, published in February 1968, by C. Fairhurst.

This report, The Theory of Stress Determination by Means of Stress Relief Techniques in a Transversely Isotropic Medium, by Dr. Berry, presents the analytical expressions for the strains and displacements at the surface of a circular hole in an infinite transversely isotropic elastic medium due to general stresses at infinity. This solution permits the influence of rock anisotropy to be considered in the stress-relief (overcoring) techniques.

Investigation of the effect of rock anisotropy on stresses in an elastic inclusion, the theoretical basis of the remaining important class of stress-determination methods, is now in progress and will be published in a subsequent report.

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